Sec 7.4 Gallery of solutions

System $\underline{X}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{X}$, general solutions $\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2$

Can classify how solution curves look/behave near origin.

Part I : Distinct real eigenvalues:

Say λ_1, λ_2 e-values, $(\lambda_1 \neq \lambda_2)$

$\underline{v_1}, \underline{v_2}$ e-vectors, lin indep

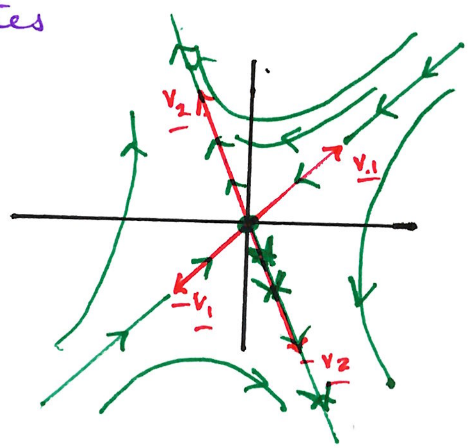
solution is $\underline{X} = c_1 \cdot \underbrace{e^{\lambda_1 t}}_{\text{decays} \to 0 \text{ if } \lambda_1 < 0} \underline{v_1} + c_2 \cdot \underbrace{e^{\lambda_2 t}}_{\text{goes to } \infty \text{ if } \lambda_2 > 0} \underline{v_2}$

Case 1 : λ_1, λ_2 nonzero, opposite sign

Ex: $\lambda_1 < 0 < \lambda_2$

· pos./neg. eigenvalue accelerates curve away/to origin

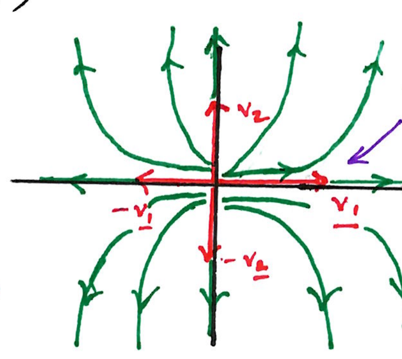
In this case, the origin is a saddle point.



Case 2 : λ_1, λ_2 have same sign $(\neq 0)$

Ex: $0 < \lambda_1 < \lambda_2$

$c_1 \underbrace{e^{\lambda_1 t}}_{\lambda_1 < \lambda_2 \Rightarrow \text{ stronger influence } t \to -\infty} \underline{v_1} + c_2 \underbrace{e^{\lambda_2 t}}_{\lambda_2 > \lambda_1 \Rightarrow \text{ stronger push as } t \to \infty} \underline{v_2}$



near O, $\underline{v_1}$ line is asymp. for curves

In this case, origin is an improper nodal source/sink.

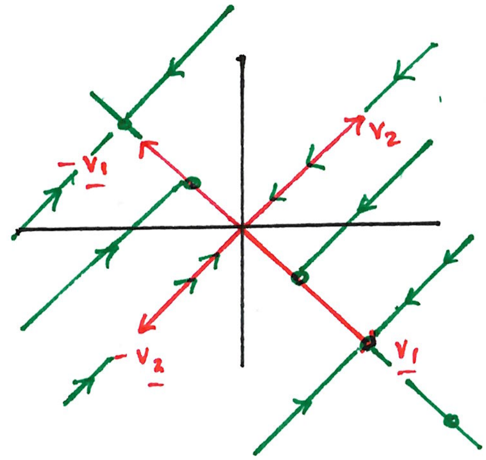
(b/c the "accelerations" in diff. directions are unequal)

(source of curves go out
sink of curves come in)

Case One eigenval $= 0$ (say, ex: $\lambda_1 = 0$, $\lambda_2 = -3$)

no movement along $\underline{v_1}$ line

we get parallel lines



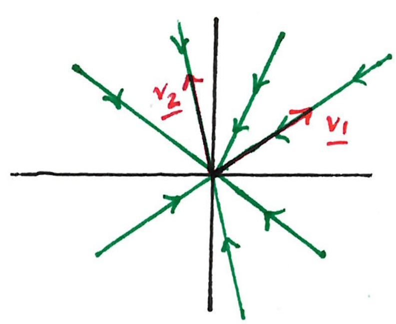
PART II : Repeated real eigenvalue :

Case : eigenvalue $\lambda \neq 0$ is repeated (has alg. mult. 2) and is complete (defect $= 0$) $\Longrightarrow$ $\underline{v_1}, \underline{v_2}$ are lin. indep. eigenvectors for λ. ("true eigenvectors")

Solution is $\underline{X} = c_1 e^{\lambda t} \underline{v_1} + c_2 e^{\lambda t} \underline{v_2} = e^{\lambda t}(c_1 \underline{v_1} + c_2 \underline{v_2})$

Ex: $\lambda = -1$, $\underline{v_1}, \underline{v_2}$ are basis for $\mathcal{E}_{-1}$:

Here the origin is a proper nodal source/sink



Case $\lambda \neq 0$ repeated, but is defective (alg. mult. 2, only "defective option" is defect $= 2-1 = 1$.)

have $\underline{v_1}$ "true eigenvector"

$\underline{v_2}$ "virtual eigenvector"

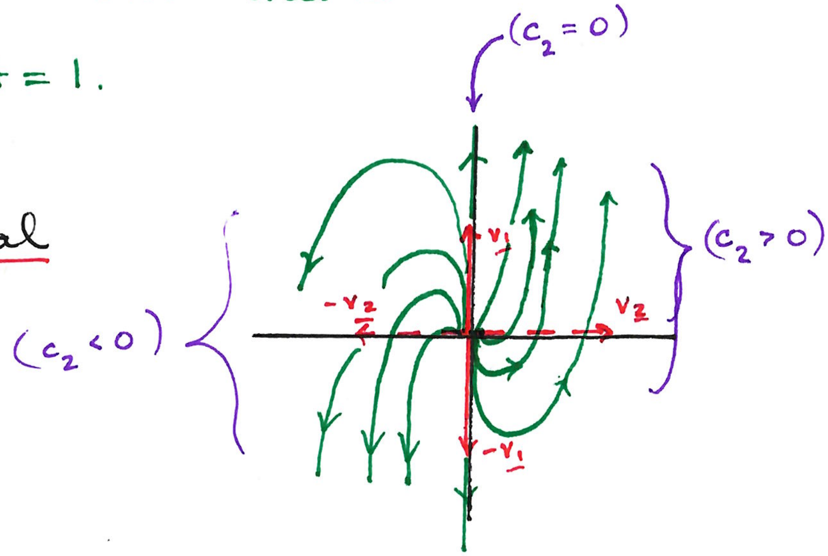
Solution is $\underline{X} = c_1 e^{\lambda t} \underline{v_1} + c_2 e^{\lambda t}(t\underline{v_1} + \underline{v_2})$

$$= \underbrace{(c_1 + c_2 t)e^{\lambda t}}_{\text{eventually dominates}} \underline{v_1} + c_2 \underbrace{e^{\lambda t}}_{\text{over this}} \underline{v_2}$$

eventually dominates over this

Ex: $\lambda > 0$, defect $= 1$.

Here, again origin is an improper nodal source/sink



☆ Read other cases in section 7.4, and look at summary table there.

☆ For complex eigenvalues, can't draw eigenvectors.